Bézout, Hankel, and Loewner Matrices

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ABSTRACT

Relations between the classes of Bézout, Hankel, and Loewner matrices and of their inverses are investigated. All these classes appear as matrix realizations of one operator in suitable bases. Different bases in the space of all polynomials of degree not exceeding $n - 1$ are considered. These bases correspond to different expansions of the polynomial

$$\frac{w(z)p(x) - p(z)w(x)}{z - x}.$$

We introduce a matrix K corresponding to this polynomial; this matrix together with its limit forms may be used to represent all the classes of matrices mentioned. In particular we obtain a new interpretation of the variant of generalized Loewner matrices introduced recently by the second-named author.

INTRODUCTION

The idea of using a compression of the shift for the construction of models for operators is of fundamental importance in recent investigations in operator theory; the pioneering contribution of D. Sarason has influenced the work since 1967, when his paper appeared. In the finite dimensional case P. Fuhrmann developed the theory of polynomial models for applications in systems theory. The statement of many results important for applications

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in systems theory becomes particularly transparent if the operators involved are realized as matrices in special bases; for this purpose Fuhrmann introduces suggestive names for some natural bases in the space of polynomials (control basis, spectral basis, etc.). The role of the shift is played, in this theory, by the operator M_p acting on polynomials as multiplication by the independent variable modulo p.

The notion of Bézout matrices arose in a natural manner from elimination problems for a system of two algebraic equations in two unknowns. The Bézout matrix corresponds to a pair of polynomials (w, p, say), and its determinant possesses the resultant property. If $C(p)$ stands for the companion matrix of p, then clearly the determinant of $w(C(p))$ also possesses the resultant property. Thus it is not surprising that the Bézout matrix of the pair of polynomials w, p is closely related to $w(C(p))$. The relation between these two matrices is given by the important Barnett formula [2]. In Proposition 2.2 we clarify the connection of the Barnett formula and the following important fact: The Bézout matrix is the matrix of the operator $w(M_p)$ in suitable bases. This fact was stated first by P. Fuhrmann (see, for instance, [6], [7], or [8]).

Our starting point is the observation that the Barnett formula is equivalent to the fact that the Bezoutian matrix is just a realization of the operator $w(M_p)$. We show that the same operator will do for several other important classes of matrices including Hankel and Loewner matrices: these matrices appear as matrices of this operator in suitably chosen natural bases. This sheds new light on and provides an interesting explanation of the well-known relationship between these matrices. As a by-product we obtain a proof of the classical theorem of Lander. Our considerations also provide an explanation of the well-known relationship between important classes of matrices (such as Hankel and Loewner) by showing that these matrices, if corresponding to a given pair of polynomials w, p, are realizations of one and the same operator $w(M_p)$.

We describe a new class of matrices (called K in the sequel) as a natural addition to the classical families of special matrices. The Bezoutian matrices appear as particular cases of matrices K. We identify bases with respect to which the operator $w(M_p)$ appears as Bézout, as the matrix K, and as Hankel, Loewner, generalized Loewner, and inverse Loewner. As a by-product we obtain a proof of the classical theorem of Lander.

There is yet another way of interpreting these results; to this end it is convenient to slightly extend the meaning of the term generating function. Given a matrix M and two systems of polynomials $g_0, \dots, g_{n-1}$ and $h_0, \dots, h_{n-1}$, it is possible to consider the polynomial $\sum m_{ik} h_i(y) g_k(z)$ obtained by replacing the monomials $1, z, \dots, z^{n-1}$ with the polynomials g and h. In this manner the above expression appears as a natural extension of the classical notion of a generating function. Using this terminology, our results assume the following form: we consider a fixed pair of polynomials p, w; to each class of matrices (Bézout, Hankel, Loewner, etc.) there corresponds a pair $\{g_0, \dots, g_{n-1}\}$ and $\{h_0, \dots, h_{n-1}\}$ in such a manner that the generating function in this extended sense equals

$$R(y, z) = \frac{p(y)w(z) - w(y)p(z)}{y - z}.$$

We introduce a new class of matrices (called QB matrices in the sequel) as a natural generalization of Bézout matrices. They are obtained in a natural manner from the polynomial $R(y, z)$ as follows: we choose two sets of interpolation nodes $\{y_0, \dots, y_{n-1}\}$ and $\{z_0, \dots, z_{n-1}\}$, and define the corresponding QB matrix as

$$R(y_i, z_j), \qquad i, j = 0, \dots, n-1.$$

The i, jth entry of the matrix is thus

$$\frac{p(y_i)w(z_j) - w(y_i)p(z_j)}{y_i - z_j}.$$

Let us remark that a particular case of QB matrices (for distinct interpolation nodes) was investigated, in another context, by Heinig and Rost.

Of course, the interpolation nodes need not be distinct—in that case we consider higher difference quotients in an obvious manner. One important particular case is the classical Bézout matrix resulting from the choice of the point zero as a single interpolation node of multiplicity n. It consists thus of the coefficients of the polynomial R—in other words, given a matrix M of type (n, n) and a system of polynomials $g_0, \dots, g_{n-1}$, it is possible to consider a polynomial in two variables $\Sigma m_{ik} g_i(y) g_k(z)$ obtained by replacing the monomials $1, z, \dots, z^{n-1}$ with $g_0(z), \dots, g_{n-1}(z)$. Our results assume the following form: Given a pair of polynomials p, w, to each class of matrices (Bézout, Hankel, Loewner, etc.) there corresponds a system of polynomials in such a manner that the generating function in this extended sense equals

$$R(y, z) = \frac{p(y)w(z) - w(y)p(z)}{y - z}.$$

The Bezoutian matrix is a particular case, since it is generated by $R(y, z)$ in the classical sense. It consists of the coefficients of the polynomial R, in other words,

$$\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i\,\partial x^j}R(z, x)\Big|_{z=0,\,x=0}, \qquad i, j = 0, \ldots, n-1.$$

1. PRELIMINARIES

Throughout the paper p will be a fixed polynomial of degree n, $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ with $p_n \neq 0$. The companion matrix $C(p)$ of the polynomial p is defined as

$$C(p) = \begin{pmatrix} 0 & \cdots & 0 & -p_0/p_n \\ 1 & \cdots & 0 & -p_1/p_n \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -p_{n-1}/p_n \end{pmatrix}.$$

Polynomials of degree at most $n - 1$ will be identified with the corresponding (column) vectors of their coefficients.

For every complex z we shall denote by $\pi(z)$ the row vector

$$\pi(z) = (1, z, \ldots, z^{n-1}).$$

Thus the value of a polynomial f at the point z will be expressed as the scalar product $\pi(z)f$.

Let T be a linear operator from a vector space E_1 into a vector space E_2. If R is a row of vectors $a_j \in E_1$,

$$R = (a_0, \ldots, a_{n-1}),$$

we denote by TR the row (in E_2)

$$TR = (Ta_0, \ldots, Tn_{m-1}).$$

Let $B_1 = (b_{1,0}, \ldots, b_{1,n-1})$ be a basis of E_1, and $B_2 = (b_{2,0}, \ldots, b_{2,m-1})$ be a basis of E_2. The matrix of T in the bases B_1 and B_2, denoted by

$\mathscr{M}(T; B_1, B_2)$, is defined by the relation

$$TB_1 = B_2\mathscr{M}(T; B_1, B_2).$$

If W_1 and W_2 are regular matrices of type (n, n) and (m, m) respectively and if new bases B_1' and B_2' are introduced by the relations $B_1' = B_1W_1$, $B_2' = B_2W_2$, then

$$\mathscr{M}(T; B_1', B_2') = W_2^{-1}\mathscr{M}(T; B_1, B_2)W_1.$$

The following observation will be useful:

LEMMA 1.1. *Suppose B_1 and B_2 are two bases of a vector space E, and let T be a linear operator in E. If W is the matrix for which $B_1 = B_2W$, then*

$$\mathscr{M}(T; B_2, B_1) = W^{-1}\mathscr{M}(T; B_1, B_2)W^{-1}.$$

Indeed, $TB_2 = TB_1W^{-1} = B_2\mathscr{M}(T; B_1, B_2)W^{-1} = B_1W^{-1}\mathscr{M}(T; B_1, B_2)W^{-1}$.

We define the flip matrix J and the shift matrix S_n as follows:

$$J = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad S_n = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

An array h of interpolation nodes will be a sequence $h_0, \ldots, h_{l-1}$ of l distinct complex numbers and a sequence of positive integers $k_0, \ldots, k_{l-1}$ such that $k_0 + \cdots + k_{l-1} = n$.

The Vandermonde matrix $V(h)$ corresponding to the array h is defined as follows:

$$V(h) = \begin{pmatrix} V(h_0, k_0) \\ \vdots \\ V(h_{l-1}, k_{l-1}) \end{pmatrix}.$$

Here $V(h_j, k_j)$ stands for the (k_j, n) matrix

$$V(h_j, k_j)_{ik} = \binom{k}{i} h_j^{k-i} \qquad \text{for} \quad 0 \leqslant i \leqslant k_j - 1, \quad 0 \leqslant k \leqslant n - 1.$$

Now consider another polynomial w of degree at most n, and denote by R the polynomial

$$R(z, x) = \frac{w(z)p(x) - p(z)w(x)}{z - x}.$$

Let $B(w, p)$ be the matrix the generating function of which is R; thus

$$\pi(z)B(w, p)\pi(x)^T = R(z, x).$$

In the particular case $B(p, 1)$ we have

$$\pi(z)B(p, 1)\pi(x)^T = Q(z, x),$$

where Q is the difference quotient

$$Q(z, x) = \frac{p(z) - p(x)}{z - x}.$$

Clearly $B(p, 1) = J\hat{p}(S_n)$, where $\hat{p}$ is the reciprocal polynomial $\hat{p}(z) = z^n p(1/z)$.

We denote by S the shift operator on the (infinite dimensional) vector space of all polynomials:

$$(Sf)(z) = zf(z).$$

The backward shift S^* assigns to a polynomial g the polynomial

$$(S^*g)(z) = \frac{g(z) - g(0)}{z}.$$

Denote by P_j the polynomial $S^{*j+1}p$, and let P stand for the basis

$$P = (P_0, \dots, P_{n-1})$$

in the space of all polynomials of degree at most $n - 1$. If we identify the P_j's with the (column) vector of their coefficients, we obtain the identity

$$P = (P_0, \dots, P_{n-1}) \sim B(p, 1) = J\hat{p}(S_n).$$

Let us observe that

$$\pi(z) P \pi(x)^T = Q(z, x).$$

We denote by $e_0, \dots, e_{n-1}$ the standard basis of C^n taken as coefficient vectors of polynomials. In this manner

$$E = (e_0, \dots, e_{n-1})$$

represents the identity matrix and

$$\pi(z) E = (1, z, \dots, z^{n-1}).$$

The following notation will be used for the Hankel and Loewner matrix corresponding to the rational function f/g, $\deg f, \deg g \leqslant n$: If a, b are arrays of interpolation nodes $a_i, b_i, i = 0, \dots, n-1$, then

$$L_{a,b}(f/g) = \left(\frac{f(a_i)/g(a_i) - f(b_j)/g(b_j)}{a_i - b_j} \right)_{i,j=0,\dots,n-1}.$$

If $\deg f < \deg g$, then

$$H(f/g) = (\alpha_{i+j})_{ij},$$

where

$$x^{2n} \left(\frac{f(x)}{g(x)} - \sum_0^{2n-2} \frac{\alpha_k}{x^{k+1}} \right)$$

is a function bounded at infinity.

2. BÉZOUT MATRICES

We begin by recalling an identity for the operator S [11].

PROPOSITION 2.1. *The following identity holds*:

$$(x - S)\sum_{0}^{n-1} x^k S^{*k+1} = x^n S^{*n} - SS^* + (1 - SS^*)\sum_{1}^{n-1} x^k S^{*k}.$$

When applied to the polynomial p it yields

$$(x - S)\sum_{0}^{n-1} x^k P_k = p(x) - p.$$

This identity is nothing more than the operator version of a classical fact. For the convenience of the reader we repeat the proof.

Proof. We have

$$(x - S)\sum_{0}^{n-1} x^k S^{*k+1} = \sum_{0}^{n-1} x^{k+1} S^{*k+1} - SS^* \sum_{0}^{n-1} x^k S^{*k}$$

$$= x^n S^{*n} - SS^* + (1 - SS^*)\sum_{1}^{n-1} x^k S^{*k}.$$

Applying the identity to the polynomial p, we obtain

$$(x - S)\sum_{0}^{n-1} x^k P_k = x^n p_n - SS^* p + \sum_{1}^{n-1} x^k (1 - SS^*) S^{*k} p$$

$$= p(x) - p.$$

∎

Denote by M_p the operator, on the space of all polynomials of degree at most $n - 1$, defined as multiplication by z modulo p. More precisely, if f is a polynomial of degree at most $n - 1$, then $M_p f$ will be the remainder, upon dividing by p, of the polynomial Sf. Clearly $M_p E = EC(p)$; the matrix of M_p in the standard bases is $C(p)$.

In the rest of this paper T will stand for the operator $w(M_p)$.

We begin with a simple proposition describing a representation of T. The first part of this proposition is obvious and folklore. The second assertion is known as the Barnett formula. It was first obtained by S. Barnett by direct matrix methods [2].

PROPOSITION 2.2. *The Bézout matrix $B(w, p)$ is the matrix of T in the bases P, E. The Bézout matrix satisfies*

$$B(w, p) = w(C(p))B(p,1).$$

Proof. The identity

$$w(z)[p(x) - p(z)] = w(z)p(x) - p(z)w(x) + p(z)[w(x) - w(z)]$$

implies

$$w(z)Q(x, z) = R(z, x) \bmod p(z).$$

Thus

$$w(z)\sum_{0}^{n-1} P_j(z)x^j = \pi(z)B(w, p)\pi(x)^T \bmod p(z),$$

in other words,

$$\pi(z)\sum_{0}^{n-1} TP_j x^j = \pi(z)B(w, p)\pi(x)^T,$$

$$TP = EB(w, p).$$

This proves the first assertion. To prove the second, we observe that

$$M_p E = EC(p)$$

implies that

$$TE = w(M_p)E = Ew(C(p)),$$

whence

$$TP = TEB(p, 1) = Ew(C(p))B(p, 1).$$

■

The Bézout matrix is, in fact, nothing more than the matrix of the coefficients in the expansion of the function $R(z, x)$ in a power series at the point $(0, 0)$. The coefficient of $z^i x^j$ equals

$$\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i \partial x^j}R(z, x)\bigg|_{z=0,\ x=0}.$$

It is possible to consider more general expansions and the corresponding matrices of coefficients; in this manner the Bézout matrix will appear as a particular case of a more general matrix.

3. INTERPOLATION

For simplicity, let us begin with the case of simple interpolation nodes. Suppose we are given an array

$$a = (a_0, \ldots, a_{n-1})$$

of n distinct numbers a_j and an array

$$b = (b_0, \ldots, b_{n-1});$$

again, we suppose that the b_j are all distinct. Denote by $V(b)$ the Vandermonde matrix

$$V(b)_{ik} = b_i^k, \qquad 0 \leqslant i, j \leqslant n - 1,$$

and let $V(a)$ be the Vandermonde matrix corresponding to a. We write $\Phi(b)$ and $\Phi_i(b)$ for the polynomials

$$\Phi(b)(x) = (x - b_0) \cdots (x - b_{n-1}),$$

$$\Phi_i(b)(x) = \frac{\Phi(b)(x)}{x - b_i}.$$

The set $\{\Phi_i(b)\}$ can be naturally called the interpolation basis corresponding to the array b. Let $\tilde{V}(b)$ be the matrix whose jth column consists of the coefficients of $\Phi_j(b)$, so that

$$\tilde{V}(b)_{ij} = \frac{1}{i!}\Phi_j(b)^{(i)}(0).$$

Observe that

$$V(b)\tilde{V}(b) = \mathrm{diag}(\Phi_i(b)(b_i)),$$

so that

$$V(b)^{-1} = \tilde{V}(b)\,\mathrm{diag}\big(\Phi_i(b)(b_i)^{-1}\big).$$

The matrices $V(a)$ and $\tilde{V}(a)$ corresponding to a are constructed in an analogous manner from a.

We have

$$(P_0,\dots,P_{n-1})V(a)^T = (Q(\cdot,a_0),\dots,Q(\cdot,a_{n-1})).$$

It follows that

$$\begin{aligned} T(Q(\cdot,a_0),\dots,Q(\cdot,a_{n-1})) &= TPV(a)^T = EB(w,p)V(a)^T \\ &= EV(b)^{-1}V(b)B(w,p)V(a)^T \\ &= \left(\frac{\Phi_0(b)}{\Phi_0(b)(b_0)},\dots,\frac{\Phi_{n-1}(b)}{\Phi_{n-1}(b)(b_{n-1})}\right)K, \end{aligned}$$

where K is the QB matrix

$$K = V(b)B(w,p)V(a)^T,$$

$$K_{ij} = \frac{w(b_i)p(a_j) - p(b_i)w(a_j)}{b_i - a_j}.$$

In this manner we have obtained the following proposition:

PROPOSITION 3.1. *Consider, in the domain space, the basis*

$$Q(\cdot, a_0), \dots, Q(\cdot, a_{n-1}),$$

and in the image space the interpolation basis corresponding to the array b, suitably normed:

$$\frac{\Phi_0(b)}{\Phi_0(b)(b_0)}, \dots, \frac{\Phi_{n-1}(b)}{\Phi_{n-1}(b)(b_{n-1})}.$$

Then the matrix of T equals K.

This fact may be restated in the form of an assertion about the coefficients of $R(z, x)$ in the expansion in the points b_i, a_j. Indeed,

$$\begin{aligned} R(z, x) &= \pi(z) B(w, p) \pi(x)^T \\ &= \pi(z) V(b)^{-1} V(b) B(w, p) V(a)^T V(a)^{-T} \pi(x)^T \\ &= \left(\frac{\Phi_0(b)(z)}{\Phi_0(b)(b_0)}, \dots, \frac{\Phi_{n-1}(b)(z)}{\Phi_{n-1}(b)(b_{n-1})} \right) K \\ &\quad \times \left(\frac{\Phi_0(a)(x)}{\Phi_0(a)(a_0)}, \dots, \frac{\Phi_{n-1}(a)(x)}{\Phi_{n-1}(a)(a_{n-1})} \right)^T. \end{aligned}$$

4. INTERPOLATION WITH MULTIPLICITIES

The next step consists in extending the considerations of the preceding section to the case of arrays with multiplicities.

Let us consider an array h of nodes with multiplicities as in section 1. We shall connect with such an array the following polynomials:

$$\Phi(h)(x) = \prod_{i=0}^{l-1}(x-h_i)^{k_i} \qquad [\deg \Phi(h) = n],$$

$$\Phi_i(h)(x) = \frac{\Phi(h)(x)}{(x-h_i)^{k_i}}, \qquad i = 0,\dots,l-1,$$

$$\Phi_{ij}(h)(x) = \Phi_i(h)(x)(x-h_i)^j, \qquad j = 0,\dots,k_i-1.$$

The Vandermonde matrix $V(h)$ was introduced in Section 1. Let us denote by $\tilde{V}(h)$ the matrix determined by the relation

$$\pi(x)\tilde{V}(h) = \big(\Phi_{0,0}(h)(x),\dots,\Phi_{0,k_0-1}(h)(x),\dots,\Phi_{l-1,0}(h)(x),\dots,\\ \Phi_{l-1,k_{l-1}-1}(h)(x)\big);$$

thus the columns of $\tilde{V}(h)$ consist of the coefficients of the polynomials $\Phi_{i,j}(h)$.

In analogy to the preceding section, we shall consider two different arrays a and b, given by the nodes $a_0,\dots,a_{r-1}$ and $b_0,\dots,b_{s-1}$ and corresponding multiplicities $m_0,\dots,m_{r-1}$ and $n_0,\dots,n_{s-1}$.

It will be necessary to compute the product

$$V(b)\tilde{V}(b).$$

If we write $\tilde{V}(b)$ in the block form

$$\tilde{V}(b) = \big(\tilde{V}(b,0),\dots,\tilde{V}(b,s-1)\big),$$

the n by n_i blocks corresponding to the interpolation nodes b_i, $i = 0,\dots,s-1$, then the product $V(b)\tilde{V}(b)$ will be a block diagonal matrix with the diagonal blocks (of size n_i)

$$V(b_i,n_i)\tilde{V}(b,i).$$

The element with indices j, k in this product is

$$\frac{1}{j!}D^j\Phi_i(b)(z)(z-b_i)^k\Big|_{z=b_i} = \frac{1}{j!}\sum_{l=0}^{j}\binom{j}{l}D^l\Phi_i(b)(z)\cdot D^{j-l}(z-b_i)^k\Big|_{z=b_i}.$$

It is evident that

$$D^{j-l}(z-b_i)^k\Big|_{z=b_i} = 0$$

for both $j-l<k$ and $j-l>k$. The whole sum reduces thus to the term corresponding to $l=j-k$, and this equals

$$\frac{1}{(j-k)!}\Phi_i(b)^{(j-k)}(b_i).$$

That means that

$$V(b_i,n_i)\tilde{V}(b,i)$$

$$= \begin{pmatrix} \Phi_i(b)(b_i) & 0 & \cdots & 0 \\ \frac{1}{1!}\Phi_i(b)^{(1)}(b_i) & \Phi_i(b)(b_i) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{(n_i-1)!}\Phi_i(b)^{(n_i-1)}(b_i) & \cdots & \frac{1}{1!}\Phi_i(b)^{(1)}(b_i) & \Phi_i(b)(b_i) \end{pmatrix}.$$

Consider the expansion of $\Phi_i(b)$ in powers of $(z-b_i)^k$ mod $(z-b_i)^{n_i}$:

$$\Phi_i(b)(z) = f_i(z-b_i).$$

In terms of f_i,

$$V(b_i,n_i)\tilde{V}(b,i) = f_i(S_{n_i}).$$

Now it is easy to invert the Vandermonde matrix $V(b)$:

$$V(b)^{-1} = \tilde{V}(b)\,\mathrm{diag}\big(g_i(S_{n_i})\big),$$

where

$$f_i(x)g_i(x) = 1 \bmod (x - b_i)^{n_i}.$$

Similarly, as in Section 3, we can write

$$TPV(a)^T = EB(w, p)V(a)^T = V(b)^{-1}V(b)B(w, p)V(a)^T.$$

The product $V(b)B(w, p)V(a)^T$ will again be denoted by K, since the QB matrix K introduced in Section 3 is a particular case of it. The entries of K are

$$\frac{1}{k!}\frac{1}{l!}\frac{\partial^{k+l}}{\partial z^k\,\partial x^l}R(z, x)\bigg|_{z=b_i,\,x=a_j}.$$

We introduce the basis

$$F = PV(a)^T:$$

$$F = (f_0, \dots, f_{n-1}) = \left(f_{0,0}, \dots, f_{0,m_0-1}, \dots, f_{r-1,0}, \dots, f_{r-1,m_{r-1}-1}\right),$$

where

$$f_{jl}(z) = \frac{1}{l!}\frac{\partial^l}{\partial x^l}Q(z, x)\bigg|_{x=a_j}.$$

Thus

$$T(f_0, \dots, f_{n-1}) = \tilde{V}(b)\,\mathrm{diag}\big(g_i(S_{n_i})\big)\,K.$$

The columns of the matrix

$$\tilde{V}(b)\,\mathrm{diag}\big(g_i(S_{n_i})\big),$$

taken as coefficients of polynomials, represent the interpolation basis for the general interpolation at the nodes b.

In this manner we have proved the following proposition:

PROPOSITION. *The matrix of T, with respect to the basis F in the domain space and the basis of interpolation polynomials in the image space, equals K.*

Note that the Bézout matrix is a particular case of this matrix K, if both arrays b and a are given by the unique node zero with multiplicity n.

In complete analogy with the preceding section, we have a parallel equivalent restatement of the result in which the matrix K appears as the matrix of coefficients upon expanding the polynomial R in interpolation bases:

$$R(z, x) = \pi(z)\tilde{V}(b)\operatorname{diag}\left(g_i(S_{n_i})\right) K \operatorname{diag}\left(g_i\left(S_{m_i}^T\right)\right)\tilde{V}(a)^T \pi(x)^T.$$

Here g_i are polynomials such that

$$f_i(x) g_i(x) = 1 \bmod (x - a_i)^{m_i},$$

and $f_i(x - a_i) = \Phi_i(a)(x)$.

5. HANKEL MATRICES

In order to establish a connection between Hankel matrices and the operator T, we shall use the identity

$$H(w/p) = B(p, 1)^{-1} B(w, p) B(p, 1)^{-1}.$$

In the case when the polynomials p and w are coprime, we can derive this identity as a simple consequence of the Barnett formula and the theorem of Lander [9]. Due to the coprimeness, there exists a polynomial f reciprocal to w modulo p. Then

$$\begin{aligned} H(w/p) &= B(f, p)^{-1} = [f(C(p)) B(p, 1)]^{-1} \\ &= B(p, 1)^{-1} f(C(p))^{-1} = B(p, 1)^{-1} w(C(p)) \\ &= B(p, 1)^{-1} B(w, p) B(p, 1)^{-1}. \end{aligned}$$

We have seen that $B(w, p)$ is the matrix of T in the bases $P = (P_0, \dots, P_{n-1})$ and $E = (e_0, \dots, e_{n-1})$:

$$TP = EB(w, p).$$

Since $P = EB(p, 1)$, it follows by Lemma 1.1 that

$$TE = P\left[B(p,1)^{-1}B(w,p)B(p,1)^{-1}\right] = PH(w/p).$$

This completes the proof for the case of coprime w and p. To extend the formula to the case of an arbitrary polynomial w it suffices to observe that every polynomial w may be arbitrarily well approximated by a polynomial $\tilde{w}$ such that $(\tilde{w}, p) = 1$.

Here is a (somewhat longer) proof that avoids the use of Lander's theorem and dispenses with the assumption $(w, p) = 1$.

The matrix $H(w/p)$ is equal to

$$(\alpha_{i+j})_{ij},$$

where

$$\sum \alpha_k x^{-k-1} = \frac{w(x)}{p(x)}.$$

Let $v = (v_0, \dots, v_{n-1})^T$ be an n-dimensional vector, and define the polynomial

$$v(x) = \pi(x)v.$$

Then evidently the ith component of the product $H(w/p)v$ is

$$(H(w/p)v)_i = \sum_{j=0}^{n-1} \alpha_{i+j}v_j,$$

and it is obviously equal to the coefficient at the power x^{-i-1} in the function

$$v(x)\cdot\frac{w(x)}{p(x)}.$$

We use this factor for $i = 0, \dots, n-1$ to see that $H(w/p)$ is the representation of the operator $M(w/p)$ (multiplication by the function w/p), followed by the projection $\hat{P}_-$ onto the space generated by $\{1/x, \dots, 1/x^n\}$, in the natural bases:

$$\hat{P}_{-}M(w/p)E = (1/x, \dots, 1/x^n)H(w/p).$$

Let us denote by Σ the operator that assigns to a vector $(h_0, \dots, h_{n-1})$ the function $\sum_0^\infty h_k x^{-k-1}$ in the following manner: The sequence $h_0, h_1, \dots$ satisfies the recurrence relation corresponding to p and has the initial conditions $h_0, \dots, h_{n-1}$. It is not difficult to verify that x^{-k} is reflected onto the function $P_{k-1}(x)/p(x)$. If we apply Σ to both sides of the above equality, we obtain

$$P_{-}M(w/p)E = (P_0/p, \dots, P_{n-1}/p)H(w/p)_{\infty \times n}.$$

Here P_{-} is the projection onto the space of infinite Laurent series with negative powers only, and

$$H(w/p)_{\infty \times n} = (\alpha_{i+j})_{i=0}^{\infty}{}_{j=0}^{n-1},$$

the series of the α_k continued by the relation

$$\alpha_k p_n + \alpha_{k-1} p_{n-1} + \cdots + \alpha_{k-n} p_0 = 0$$

for $k = 2n - 1, 2n, \dots$. It is evident that the operator $P_{-}M(w/p)$ can be equivalently replaced by $M(p^{-1})T$ [$M(p^{-1})$ means division by p]:

$$M(p^{-1})TE = (P_0/p, \dots, P_{n-1}/p)H(w/p)_{\infty \times n}.$$

Multiplying both sides by the operator $M(p)$, we obtain finally

$$TE = (P_0, \dots, P_{n-1})H(w/p).$$

PROPOSITION 5.1. *The Hankel matrix $H(w/p)$ appears as the matrix of T taken in the standard basis in the domain space and the basis P in the image space.*

Again, this fact may be restated in terms of the expansion of the function R, now in terms of the polynomials P_j:

$$R(z, x) = (P_0(z), \dots, P_{n-1}(z))H(w/p)(P_0(x), \dots, P_{n-1}(x))^T.$$

6. LOEWNER MATRICES

Let us recall the formula linking Loewner and Hankel matrices:

$$L_{b,a}(h/p) = \tilde{V}(b)^T H(f/p)\tilde{V}(a),$$

where h and f are polynomials satisfying the relation $h = -f\Phi(a)\Phi(b) \bmod p$. (See [4] and [14].) Here

$$L_{b,a}(h/p) = \left(\frac{h(b_i)/p(b_i) - h(a_j)/p(a_j)}{b_i - a_j}\right)_{i,j=0}^{n-1}.$$

We have

$$\begin{aligned}
&T(\Phi_0(a),\dots,\Phi_{n-1}(a))\\
&\quad = TE\tilde{V}(a) = PH(w/p)\tilde{V}(a)\\
&\quad = PV(b)^T \operatorname{diag}(\Phi_i(b)(b_i))^{-1}\,\tilde{V}(b)^T H(w/p)\tilde{V}(a)\\
&\quad = PV(b)^T \operatorname{diag}(\Phi_i(b)(b_i))^{-1}\,L_{b,a}(h/p)\\
&\quad = (q_0,\dots,q_{n-1})L_{b,a}(h/p), \qquad (2)
\end{aligned}$$

where

$$h = -w\Phi(a)\Phi(b) \bmod p,$$

$$q_i = \frac{Q(\cdot,b_i)}{\Phi_i(b)(b_i)}.$$

PROPOSITION 6.1. *The operator T is represented by the Loewner matrix $L_{b,a}(h/p)$, the polynomial h being given by*

$$h = -w\Phi(a)\Phi(b) \bmod p,$$

if we consider the interpolation basis connected with the array a in the domain space and the basis $(q_0,\dots,q_{n-1})$ in the image space.

The equivalent statement about the expansion of R is

$$R(z,x) = (q_0(z),\dots,q_{n-1}(z))L_{b,a}(h/p)(\hat{q}_0(x),\dots,\hat{q}_{n-1}(x))^T,$$

where

$$q_i = \frac{Q(\cdot,b_i)}{\Phi_i(b)(b_i)},$$

$$\hat{q}_i = \frac{Q(\cdot,a_i)}{\Phi_i(a)(a_i)}.$$

7. GENERALIZED LOEWNER MATRICES

In complete analogy to the preceding section, generalized Loewner matrices are related to Hankel matrices by the formula

$$L_{b,a}(h/p) = \tilde{V}(b)^T H(f/p)\tilde{V}(a)$$

(see [15]). The only difference is that here b and a respectively mean arrays of s and r interpolation nodes, $b_0,\dots,b_{s-1}$ and $a_0,\dots,a_{r-1}$, with multiplicities $n_0,\dots,n_{s-1}$ and $m_0,\dots,m_{r-1}$. The matrices $\tilde{V}(\cdot)$ assigned to given arrays were introduced in Section 4. We shall also use the polynomials $\Phi(\cdot)$ in the sense of that section. Formally, the basic relation for the general case is almost identical with the one stated in Section 6 for the case of simple interpolation nodes:

$$\begin{aligned} TE\tilde{V}(a) &= PV(b)^T \operatorname{diag}\big(g_i\big(S_{n_i}^T\big)\big)\tilde{V}(b)^T H(w/p)\tilde{V}(a) \\ &= PV(b)^T \operatorname{diag}\big(g_i\big(S_{n_i}^T\big)\big) L_{b,a}(h/p), \end{aligned}$$

where

$$h = -w\Phi(a)\Phi(b) \bmod p;$$

g_i are the polynomials introduced in Section 4.

We shall keep the notation $(q_0,\dots,q_{n-1})$ even in this more general situation:

$$(q_0, \ldots, q_{n-1}) = (q_{0,0}, \ldots, q_{0,n_0-1}, \ldots, q_{s-1,n_{s-1}-1})$$

where

$$(q_{i,0}, \ldots, q_{i,n_i-1})$$
$$= \left(Q(\cdot, b_i), \frac{1}{1!} Q'(\cdot, b_i), \ldots, \frac{1}{(n_i - 1)!} Q^{(n_i-1)}(\cdot, b_i) \right) \cdot d_i\left(S_{n_i}^T\right).$$

PROPOSITION 7.1. *The operator T is represented by the generalized Loewner matrix $L_{b,a}(h/p)$,*

$$h = -w\Phi(a)\Phi(b) \bmod p,$$

if the basis in the domain space is the general interpolation basis connected with the array a and the basis in the image space is $(q_0, \ldots, q_{n-1})$ described above.

The equivalent formulation in terms of the function R reads as follows:

$$R(z, x) = (q_0(z), \ldots, q_{n-1}(z)) L_{b,a}(h/p) (\hat{q}_0(x), \ldots, \hat{q}_{n-1}(x))^T.$$

8. INVERSE LOEWNER MATRICES

We have already used Lander's result stating that Hankel and Bézout matrices are inverse to each other. Since Loewner matrices have direct connections both to Hankel matrices (see Section 6) and to Bézout matrices (see below), it is not surprising that the operator T can be properly represented not only by Loewner but also by inverse Loewner matrices.

We use Proposition 5.1:

$$TE = PH(w/p).$$

By Lander's theorem, this can be written as

$$TE = PB(f, p)^{-1},$$

where

$$wf = -1 \bmod p$$

[the sign -1 arises from the reverse order of the polynomials p, f in $B(f, p)$]. The connection of Bézout and Loewner matrices is the following:

$$L_{a,b}(f/p) = \operatorname{diag}\left(\frac{1}{p(a_i)}\right) V(a) B(f, p) V(b)^T \operatorname{diag}\left(\frac{1}{p(b_j)}\right).$$

Thus

$$B(f, p)^{-1} = V(b)^T \operatorname{diag}\left(\frac{1}{p(b_j)}\right) L_{a,b}(f/p)^{-1} \operatorname{diag}\left(\frac{1}{p(a_i)}\right) V(a),$$

$$TE = PV(b)^T \operatorname{diag}\left(\frac{1}{p(b_j)}\right) L_{a,b}(f/p)^{-1} \operatorname{diag}\left(\frac{1}{p(a_i)}\right) V(a).$$

Multiplying by $\tilde{V}(a)$, we obtain

$$T(\Phi_0(a), \dots, \Phi_{n-1}(a))$$
$$= PV(b)^T \operatorname{diag}\left(\frac{1}{p(b_j)}\right) L_{a,b}\left(\frac{f}{p}\right)^{-1} \operatorname{diag}\left(\frac{\Phi_i(a)(a_i)}{p(a_i)}\right). \quad (2')$$

Thus

$$T\left(\frac{\Phi_0(a)}{\Phi_0(a)(a_0)} p(a_0), \dots, \frac{\Phi_{n-1}(a)}{\Phi_{n-1}(a)(a_{n-1})} p(a_{n-1})\right)$$
$$= \left(\frac{Q(\cdot, b_0)}{p(b_0)}, \dots, \frac{Q(\cdot, b_{n-1})}{p(b_{n-1})}\right) L_{a,b}(f, p)^{-1}.$$

PROPOSITION 8.1. *The operator T is represented by the inverse Loewner matrix $L_{a,b}(f, p)^{-1}$,*

$$fw = -1 \bmod p,$$

if the basis in the domain space is given by the elementary Lagrange *interpolation polynomials* $\Phi_i(a)/\Phi_i(a)(a_i)$, *normed by the constants* $p(a_i)$, *and the basis in the image space is*

$$\left\{\frac{Q(\cdot,b_0)}{p(b_0)},\dots,\frac{Q(\cdot,b_{n-1})}{p(b_{n-1})}\right\}.$$

We can also write

$$R(z,x) = \left(\frac{Q(\cdot,b_0)}{p(b_0)},\dots,\frac{Q(\cdot,b_{n-1})}{p(b_{n-1})}\right)$$
$$\times L_{a,b}(f,p)^{-1}\left(\frac{Q(\cdot,a_0)}{p(a_0)},\dots,\frac{Q(\cdot,a_{n-1})}{p(a_{n-1})}\right).$$

If we compare (2′) with (2) in Section 6, we obtain the formula expressing the inverse Loewner matrix as a product of a Loewner matrix and certain diagonal matrices, which was first given in [13]:

$$L_{a,b}(f/p)^{-1} = \operatorname{diag}\left(\frac{f(b_j)}{\Phi_j(b)(b_j)}\right) L_{b,a}\left(\frac{h}{p}\right)\operatorname{diag}\left(\frac{f(a_i)}{\Phi_i(a)(a_i)}\right).$$

Since

$$h = -w\Phi(a)\Phi(b) \bmod p$$

(Section 6), the direct connection of h and f is

$$fh = \Phi(a)\Phi(b) \bmod p.$$

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